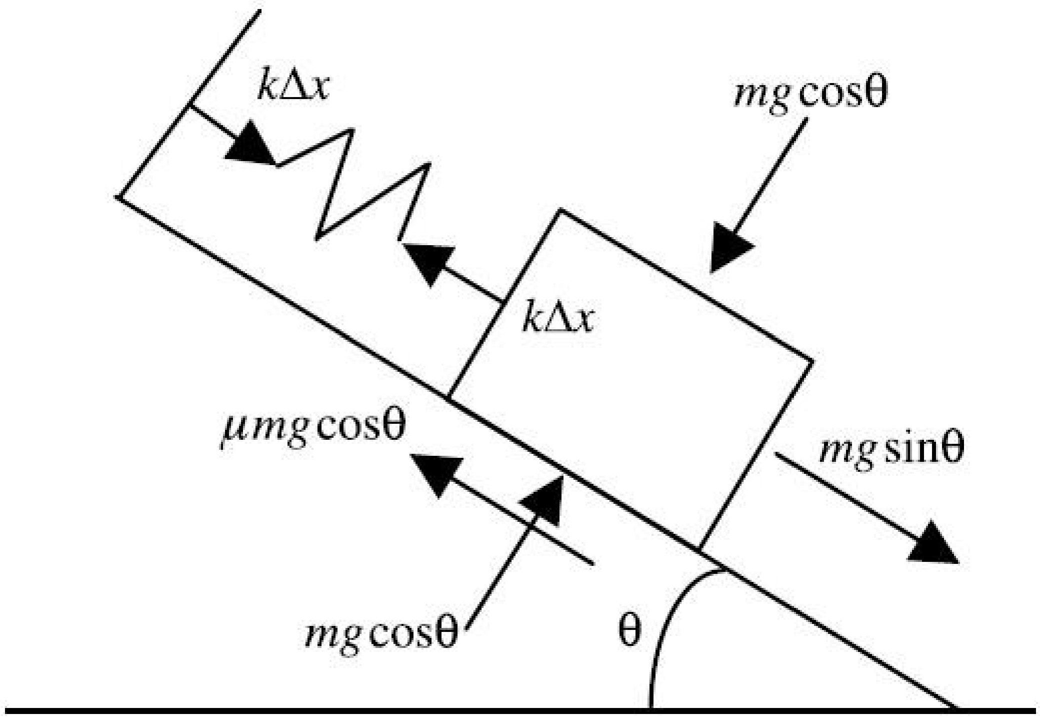
Types of Forces

- **Force balance**

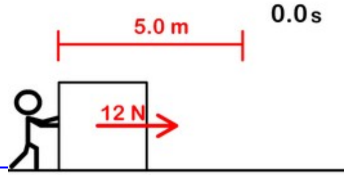


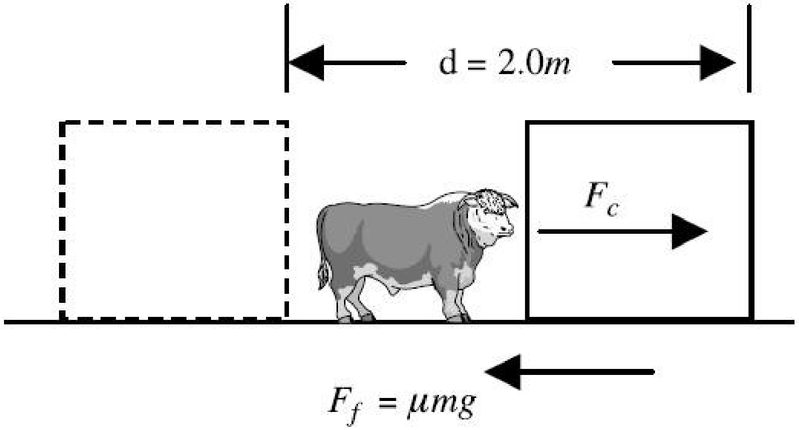
$$\sum_i \vec{F}_i = m\vec{a}$$

Energy and laws of conservation

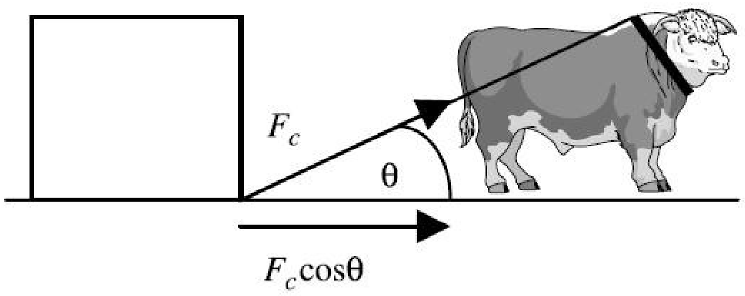
- Work
- Energy
- Power
- Rotational Motion
- Many-Particle Interactions

Work





$$\Delta W = \vec{F} \Delta \vec{r} \qquad W = \int_{r_1}^{r_2} \vec{F} d\vec{r} \qquad [\frac{\text{kg}\, m^2}{s^2}, \text{ Nm}, \ J]$$



Energy

- **Kinetic Energy**

$$E_k = \frac{mv^2}{2} = \frac{p^2}{2\mathrm{m}}$$

Kinetic energy is the energy of an object due to its motion

Energy

- **Potential Energy**

$$E_p = mgh$$

$$E_p = \frac{k \Delta x^2}{2}$$

Potential energy is the energy of an object due to its location

Energy

Thermal energy

Chemical energy

Nuclear energy

Electromagnetic energy

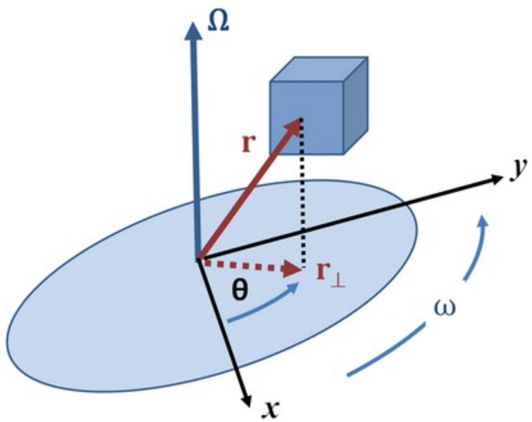
Power

$$N = \frac{\Delta W}{\Delta t} \quad [\frac{J}{s}, \text{watt}]$$

Power is defined as the amount of work performed per unit time.

Rotational Motion

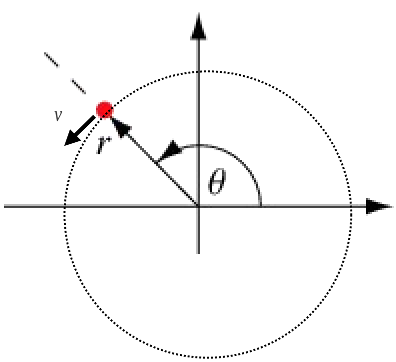
Definitions



Angular position: $\vec{\theta}(t)$

Angular velocity: $\vec{\omega}(t) = \dot{\vec{\theta}}(t)$ $[s^{-1}]$

Frequency: $f = \dfrac{\omega}{2\pi}$ $[s^{-1}]$



Angular acceleration: $\vec{\varepsilon}(t) = \dot{\vec{\omega}}(t) = \ddot{\vec{\theta}}(t)$ $[s^{-2}]$

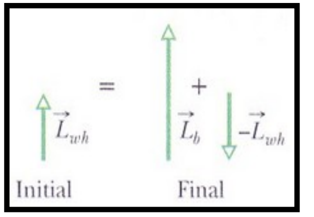
Tangential velocity: $\vec{v}(t) = [\vec{\omega}(t) \times \vec{r}(t)]$ $\left[\dfrac{m}{s}\right]$

Tangential acceleration: $\vec{a}(t) = \dfrac{d}{dt}[\vec{\omega}(t) \times \vec{r}(t)] =$

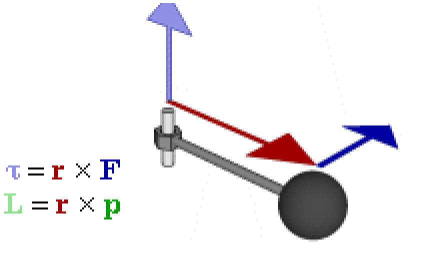
$$= [\vec{\varepsilon}(t) \times \vec{r}(t)] + [\vec{\omega}(t) \times [\vec{\omega}(t) \times \vec{r}(t)]] \quad \left[\frac{m}{s^2}\right]$$

Rotational Motion

- **Angular momentum**

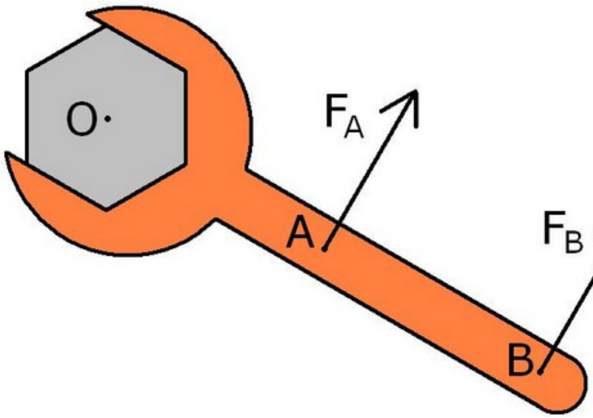


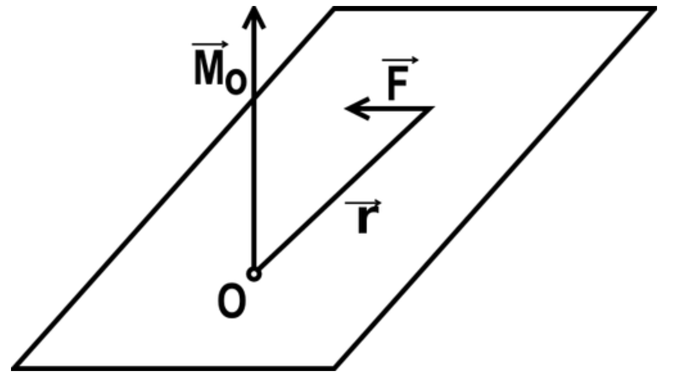
Angular momentum: $\vec{L} = [\vec{r} \times \vec{p}]$



Rotational Motion

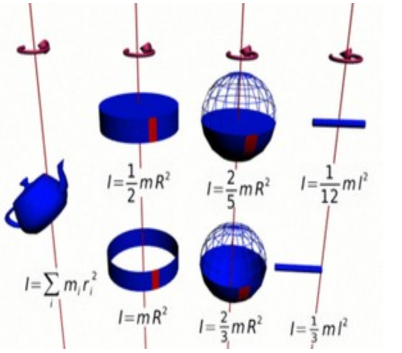
- **Torque**





Torque: $\vec{M} = \left[\vec{r} \times \vec{F}\right]$ $\left[Nm\right]$

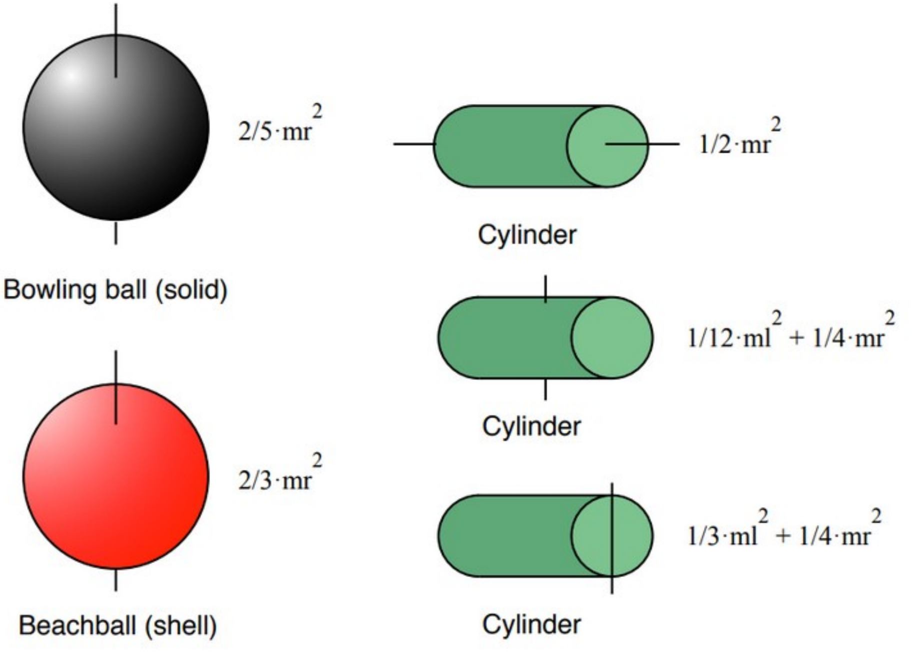
Rotational Motion



- Moment of inertia *I* [kgm²]

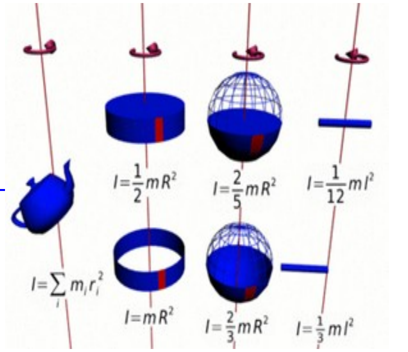
$$I=\sum_i m_i r_i^2 \;\; [\mathrm{kg\,m^2}] \qquad I=\int_V r^2 dm \;\; [\mathrm{kg\,m^2}]$$

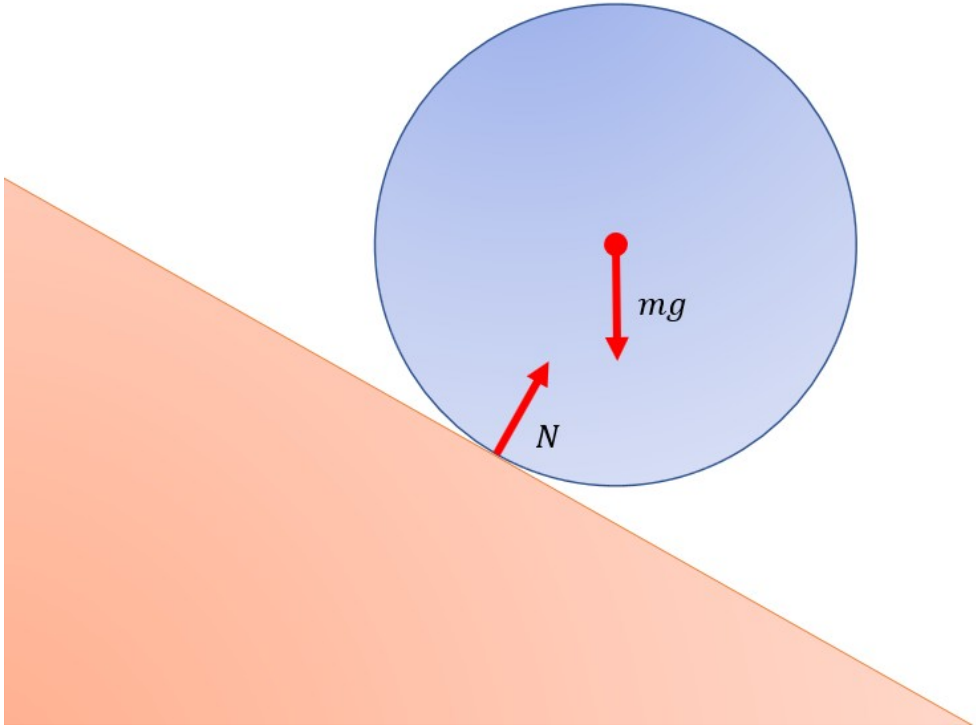
$$\vec{L}=J\,\vec{\omega} \qquad \vec{M}=J\,\vec{\varepsilon} \qquad T=\frac{I\omega^2}{2}$$



Inertia of some selected shapes

Rotational Motion



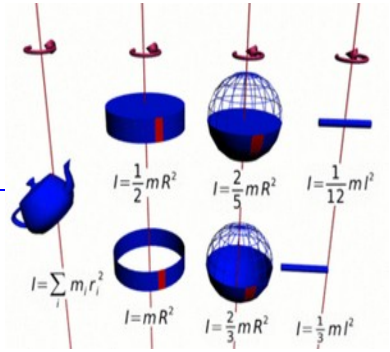


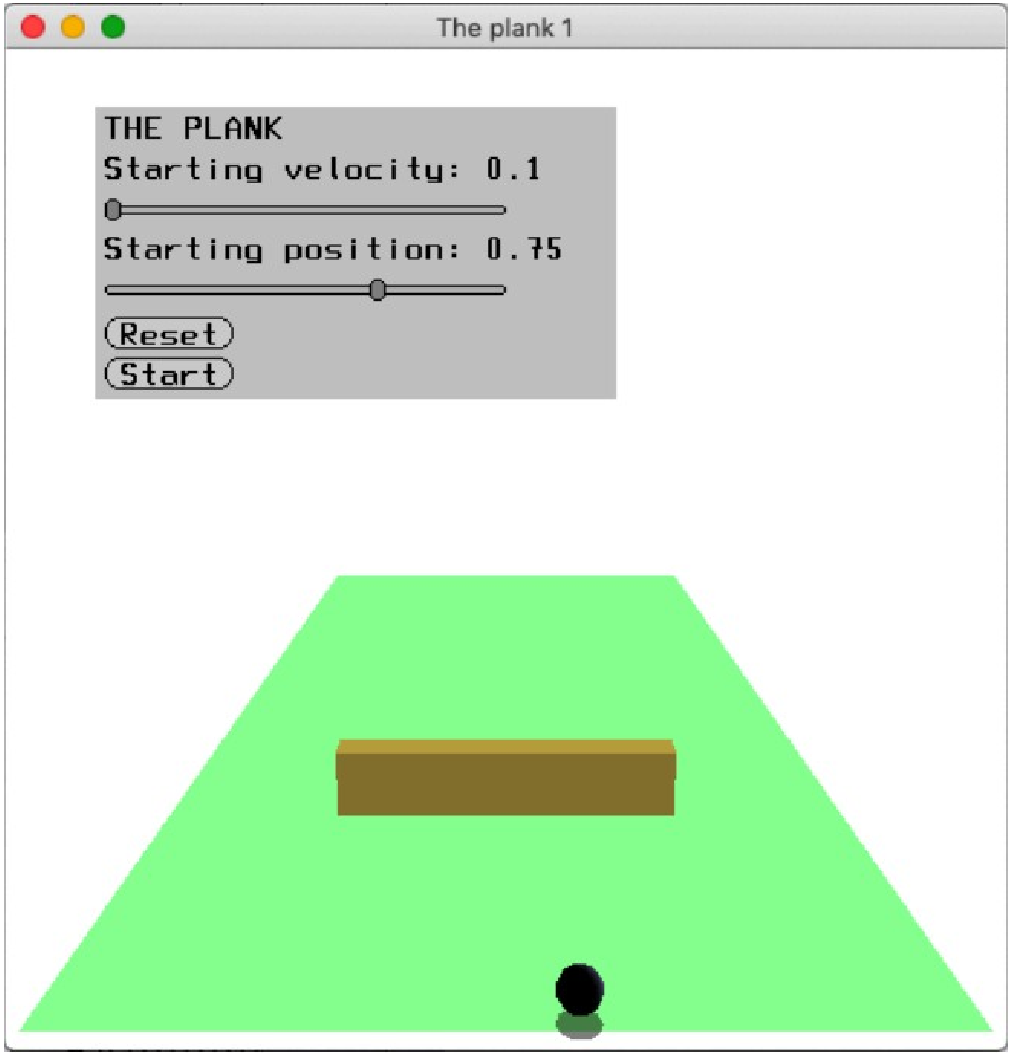
$$\vec{L}=J\,\vec{\omega}$$

$$\vec{M}=J\,\vec{\varepsilon}$$

$$T=\frac{I\omega^2}{2}$$

Rotational Motion





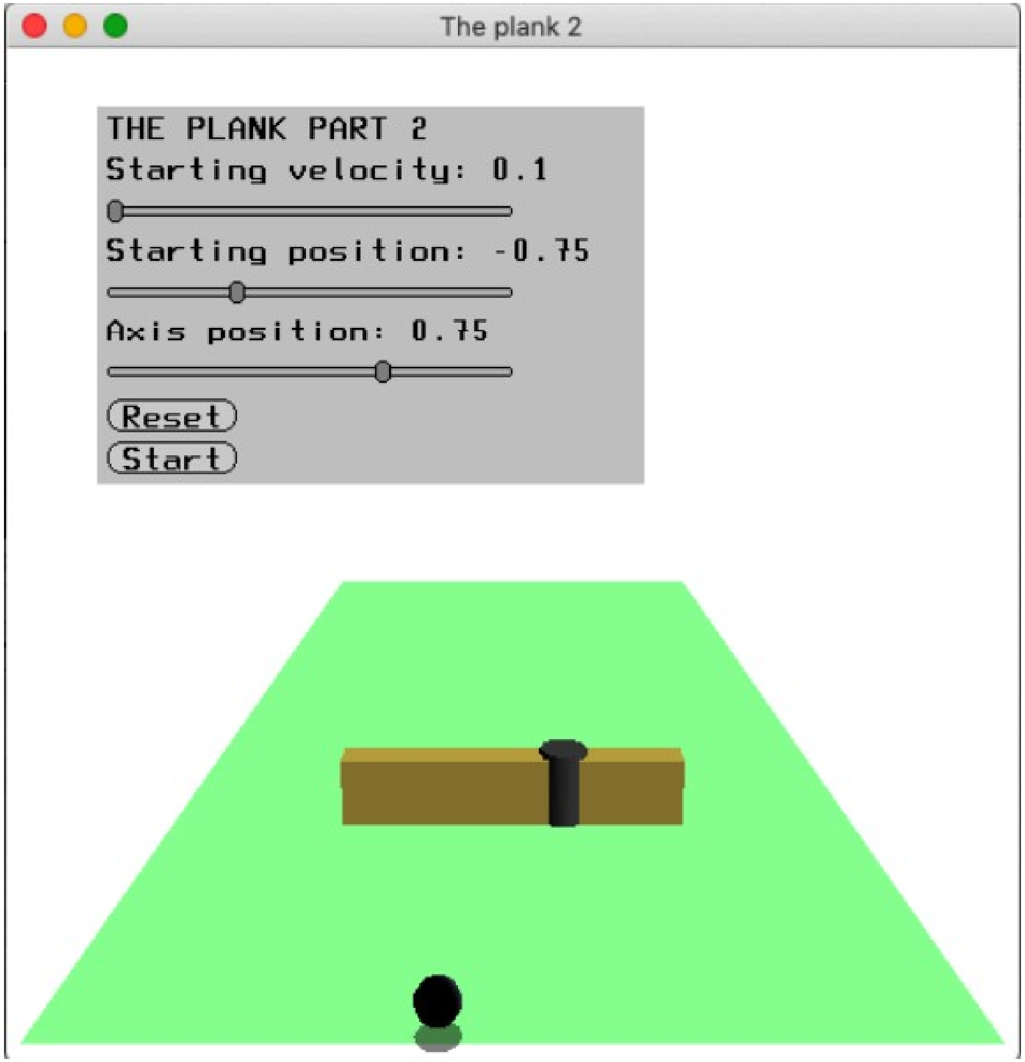
$$\vec{L} = J\,\vec{\omega}$$

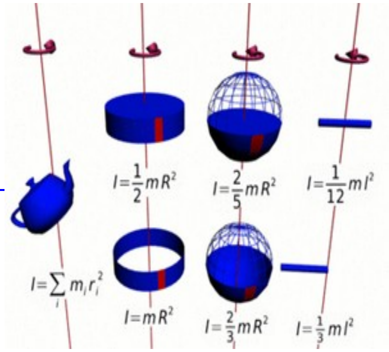
$$\vec{M} = J\,\vec{\varepsilon}$$

$$T = \frac{I\omega^2}{2}$$

https://www.computer-graphics.se/demos/files03/Beachball-demos.zip

Rotational Motion





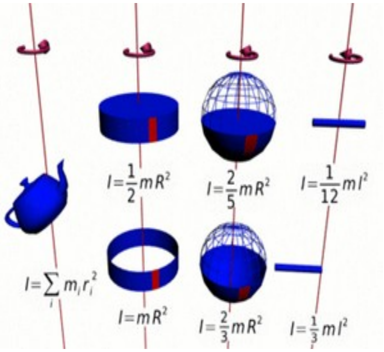
$$\vec{L}=J\,\vec{\omega}$$

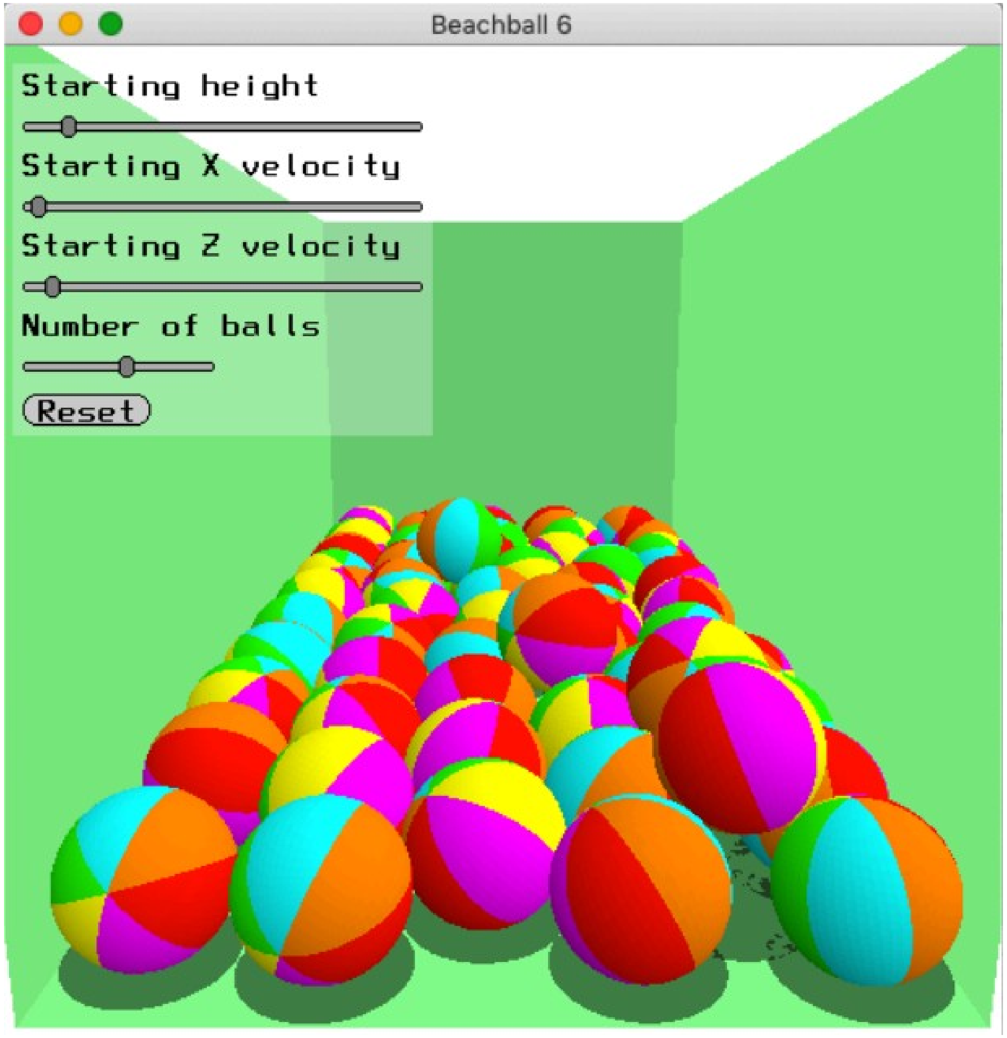
$$\vec{M}=J\,\vec{\varepsilon}$$

$$T=\frac{I\omega^2}{2}$$

https://www.computer-graphics.se/demos/files03/Beachball-demos.zip

Rotational Motion





$$\vec{L} = J\,\vec{\omega}$$

$$\vec{M} = J\,\vec{\varepsilon}$$

$$T = \frac{I\omega^2}{2}$$

https://www.computer-graphics.se/demos/files03/Beachball-demos.zip

Rotational Motion

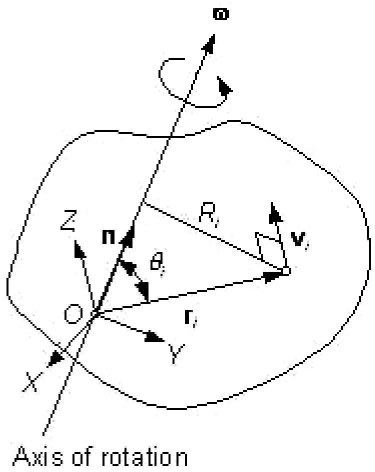
$$\vec{L} = \hat{J}\,\vec{\omega} \qquad \vec{M} = \hat{J}\,\vec{\varepsilon}$$

- Inertia tensor

$$J = \begin{bmatrix} I_{xx} & -I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & I_{yz} & I_{zz} \end{bmatrix}$$

Kinetic energy: $$T = \frac{1}{2}\vec{\omega}\,\hat{J}\,\vec{\omega}$$

Angular Momentum of a Rigid Body



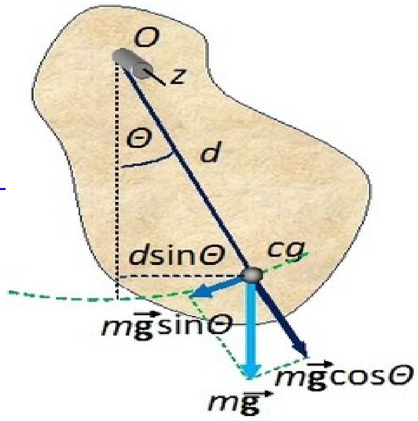
$$\vec{L}=\sum_i \vec{l}_i=\sum_i \left[\vec{r}_i\times m_i\vec{v}\right]=\sum_i m_i\left[\vec{r}_i\times\left[\vec{\omega}\times\vec{r}_i\right]\right]$$

$$=\sum_i m_i\begin{bmatrix}0 & -z_i & y_i\\ z_i & 0 & -x_i\\ -y_i & x_i & 0\end{bmatrix}\begin{bmatrix}0 & -\omega_z & \omega_y\\ \omega_z & 0 & -\omega_x\\ -\omega_y & \omega_x & 0\end{bmatrix}\begin{bmatrix}x_i\\ y_i\\ z_i\end{bmatrix}$$

$$=\sum_i m_i\begin{bmatrix}0 & -z_i & y_i\\ z_i & 0 & -x_i\\ -y_i & x_i & 0\end{bmatrix}\begin{bmatrix}z_i\omega_y-y_i\omega_z\\ x_i\omega_z-z_i\omega_x\\ y_i\omega_x-x_i\omega_y\end{bmatrix}=\sum_i m_i\begin{bmatrix}\varpi_x\left(y_i^2+z_i^2\right)-\omega_y x_i y_i-\omega_z x_i z_i\\ -\omega_x x_i y_i+\varpi_y\left(x_i^2+z_i^2\right)-\omega_z y_i z_i\\ -\omega_x z_i x_i-\omega_y y_i z_i+\varpi_z\left(x_i^2+y_i^2\right)\end{bmatrix}$$

$$=\begin{bmatrix}\left(\sum_i m_i\left(y_i^2+z_i^2\right)\right)\varpi_x+\left(-\sum_i m_i x_i y_i\right)\omega_y+\left(-\sum_i m_i x_i z_i\right)\omega_z\\ \left(-\sum_i m_i x_i y_i\right)\omega_x+\left(\sum_i m_i\left(x_i^2+z_i^2\right)\right)\varpi_y+\left(-\sum_i m_i y_i z_i\right)\omega_z\\ \left(-\sum_i m_i x_i z_i\right)\omega_x+\left(-\sum_i m_i z_i y_i\right)\omega_y+\left(\sum_i m_i\left(x_i^2+y_i^2\right)\right)\varpi_z\end{bmatrix}=\begin{bmatrix}\left(\sum_i m_i\left(y_i^2+z_i^2\right)\right) & \left(-\sum_i m_i x_i y_i\right) & \left(-\sum_i m_i x_i z_i\right)\\ \left(-\sum_i m_i x_i y_i\right) & \left(\sum_i m_i\left(x_i^2+z_i^2\right)\right) & \left(-\sum_i m_i y_i z_i\right)\\ \left(-\sum_i m_i x_i z_i\right) & \left(-\sum_i m_i z_i y_i\right) & \left(\sum_i m_i\left(x_i^2+y_i^2\right)\right)\end{bmatrix}\begin{bmatrix}\varpi_x\\ \varpi_y\\ \varpi_z\end{bmatrix}$$

Physical pendulum



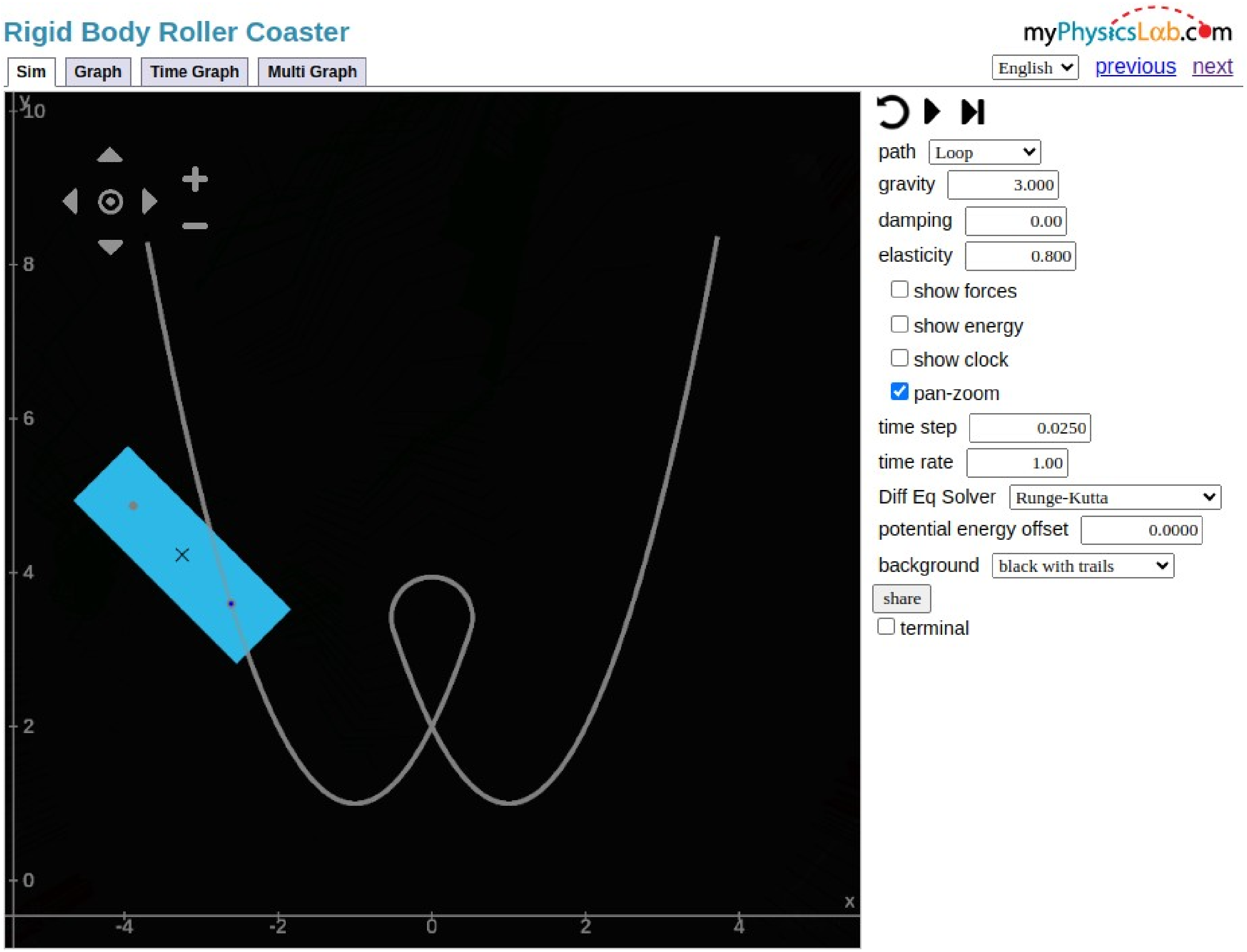
$$\vec{F} = -k\Delta\vec{x} \qquad \vec{M} = -k\Delta\vec{\theta}$$

$$\vec{M} = \hat{J}\,\ddot{\vec{\theta}}$$

$$J\ddot{\theta} + k\theta = 0$$

$$\ddot{\theta} + \varpi^2\theta = 0$$

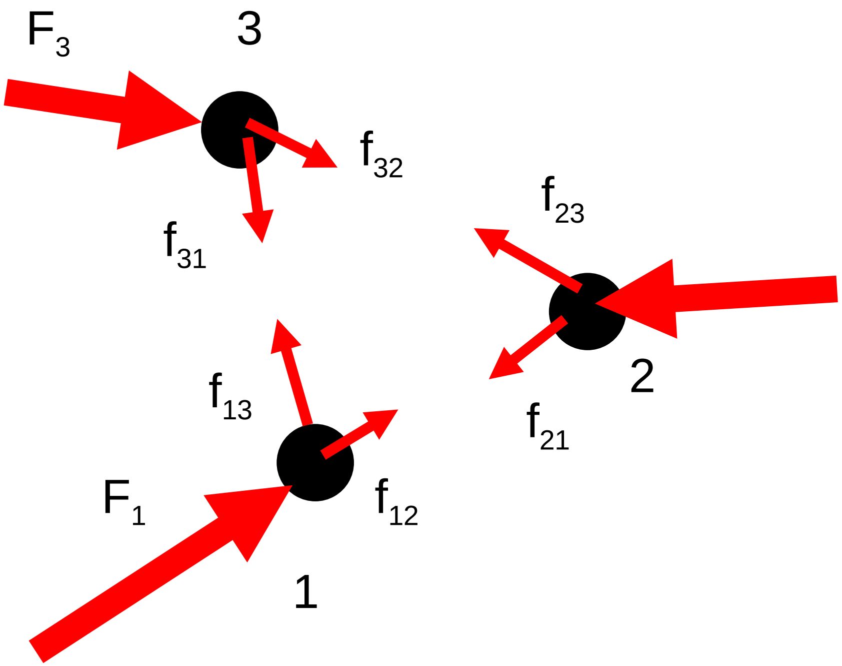
Physical pendulum



https://www.myphysicslab.com/roller/rigid-body-roller-en.html

Many-Particle Interactions

- **For an N particles system, in equilibrium**



$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i$$

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times m \vec{v}_i$$

$$\sum_i \vec{F}_i = 0$$

resultant force $= \mathbf{0}$

$$\sum_i r_i \times \vec{F}_i = 0$$

resultant moment (torque) $= \mathbf{0}$

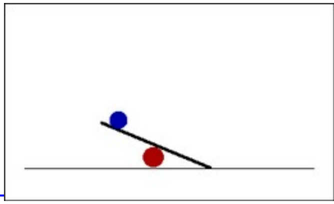
Conservation laws

$$\sum_i m_i = const$$

$$\sum_i E_i = const$$

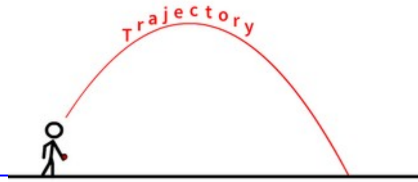
$$\sum_i \vec{P}_i = const$$

$$\sum_i \vec{L}_i = const$$



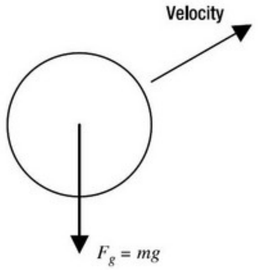
Projectiles

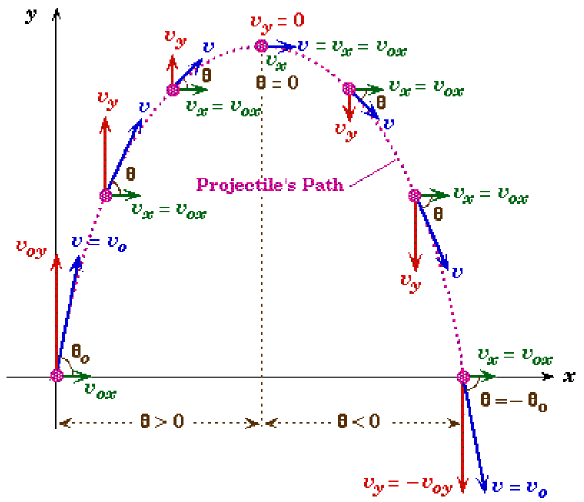
- Topics
  - The gravity-only model
  - Aerodynamic drag
  - Laminar and turbulent flow
  - Wind effects
  - Spin effects
  - Details on specific types of projectiles including bullets, cannonballs, and arrows



Projectiles

- The gravity-only model





Equation of motion:

$$\frac{d\vec{v}}{dt} = \vec{g} \quad \text{or} \quad \frac{d^2\vec{r}}{dt^2} = \vec{g}$$

Solution:

$$\vec{v} = \vec{v}_0 + \vec{g}\,t$$

$$\vec{r} = \vec{r}_o + \vec{v}_0 t + \frac{1}{2}\vec{g}t^2$$

or

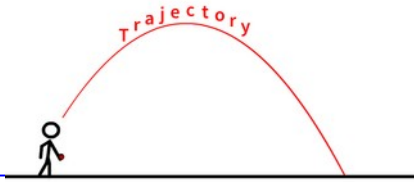
$$v_y = v_{y0} - gt,$$

$$v_x = v_{x0}$$

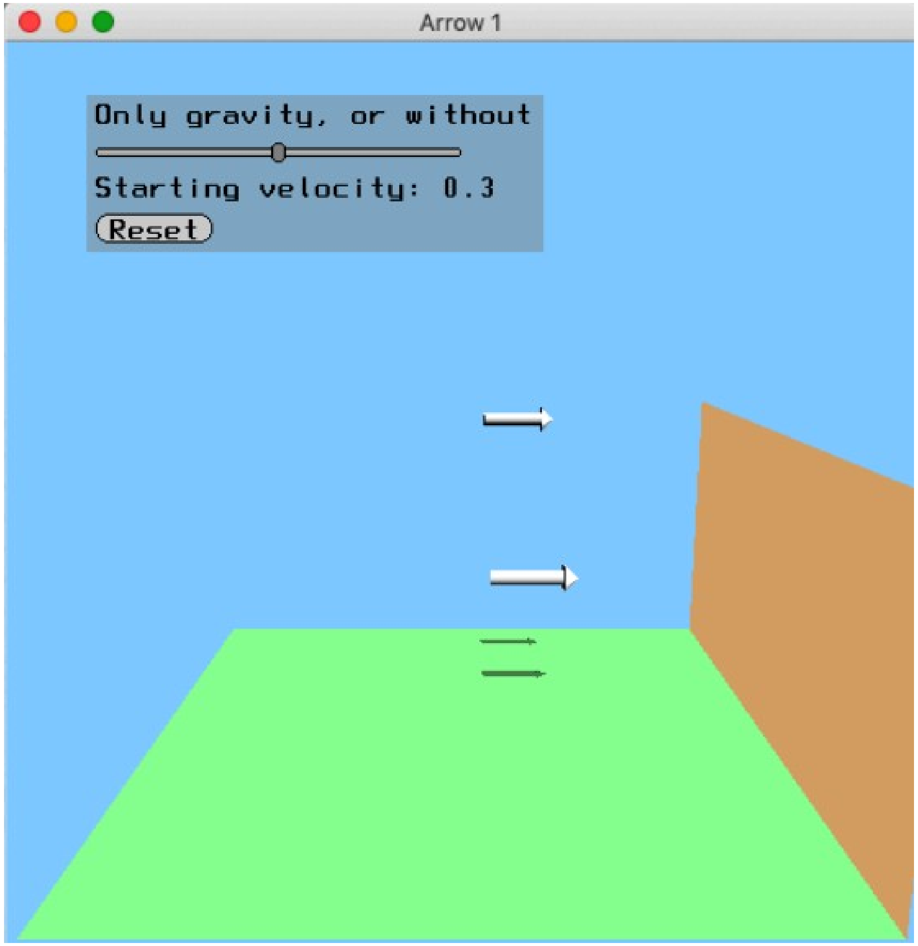
$$y = y_o + v_{y0}t - \frac{1}{2}gt^2,$$

$$x = x_o + v_{x0}t,$$

Projectiles



- The gravity-only model



Equation of motion:

$$\frac{d\vec{v}}{dt} = \vec{g} \quad \text{or} \quad \frac{d^2\vec{r}}{dt^2} = \vec{g}$$

Solution:

$$\vec{v} = \vec{v}_0 + \vec{g}\,t$$

$$\vec{r} = \vec{r}_o + \vec{v}_0 t + \frac{1}{2}\vec{g}t^2$$

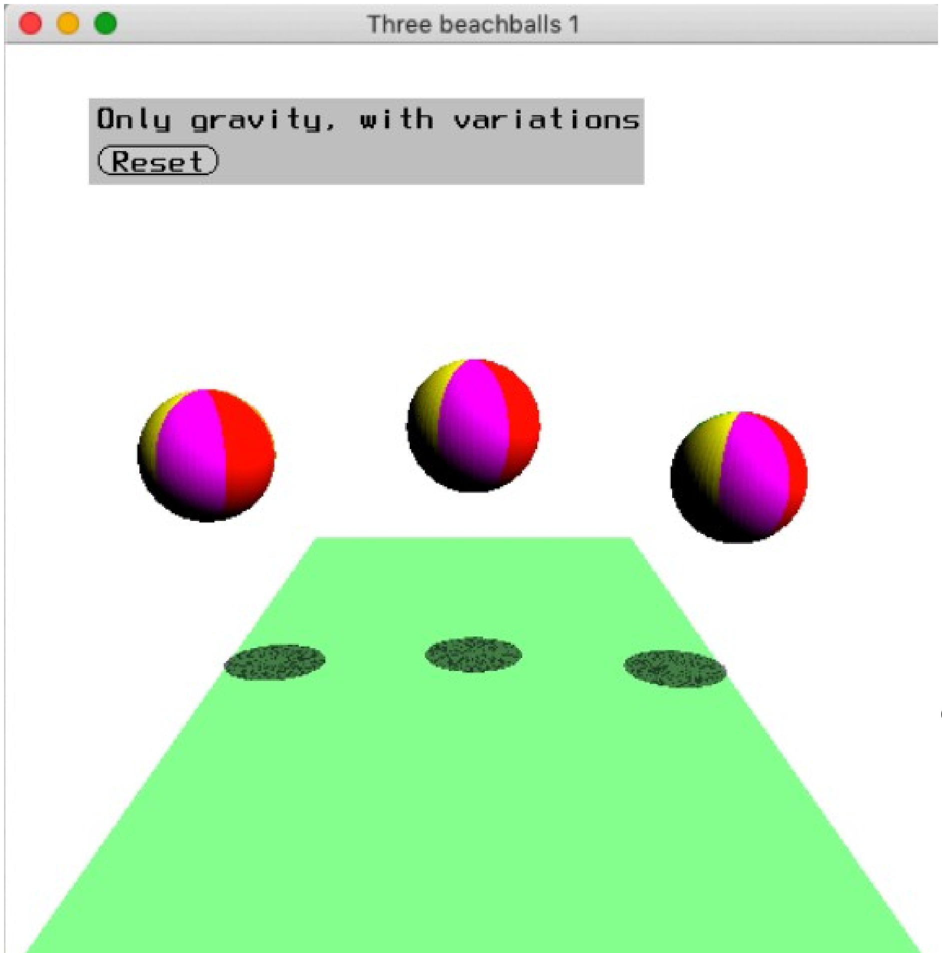
or

$$v_y = v_{y0} - gt,$$
$$v_x = v_{x0}$$
$$y = y_o + v_{y0}t - \frac{1}{2}gt^2,$$
$$x = x_o + v_{x0}t,$$

https://www.computer-graphics.se/demos/files03/Beachball-demos.zip

Projectiles

- The gravity-only model



https://www.computer-graphics.se/demos/files03/Beachball-demos.zip

Equation of motion:

$$\frac{d\vec{v}}{dt} = \vec{g} \quad \text{or} \quad \frac{d^2\vec{r}}{dt^2} = \vec{g}$$

Solution:

$$\vec{v} = \vec{v}_0 + \vec{g}\,t$$

$$\vec{r} = \vec{r}_o + \vec{v}_0 t + \frac{1}{2}\vec{g}t^2$$

or

$$v_y = v_{y0} - gt,$$

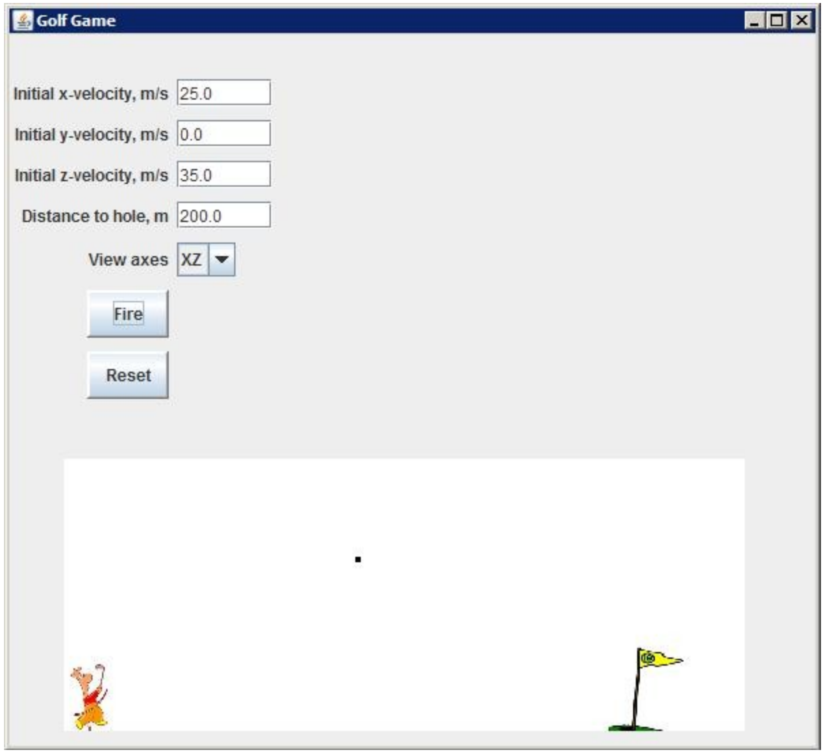
$$v_x = v_{x0}$$

$$y = y_o + v_{y0}t - \frac{1}{2}gt^2,$$

$$x = x_o + v_{x0}t,$$

Projectiles

- The gravity-only model
  - Golf Game



Projectiles

- The gravity-only model
  - Summary
    - The only force on the projectile is due to gravity, which acts in the vertical.
    - The motion in the three coordinate directions is independent.
    - The projectile trajectory is independent of mass and projectile geometry.
    - The velocity in the x- and y-directions is constant over the entire trajectory and is equal to the initial velocities in the x- and y-direction.
    - The shape of the projectile trajectory is a parabola.

Projectiles

- **Aerodynamic drag**
  - **Basic Concepts**

$$\vec{F}_{Drag} = \vec{F}_{D,\,pressure} + \vec{F}_{D,Friction}$$

Total drag

Pressure drag

Friction drag (or skin drag)

Projectiles

- **Aerodynamic drag**
  - **Drag force**

$$F_d = C_d A \rho \frac{V^2}{2}$$

Drag force

Drag coefficient

Effective area

Density of the fluid

Velocity

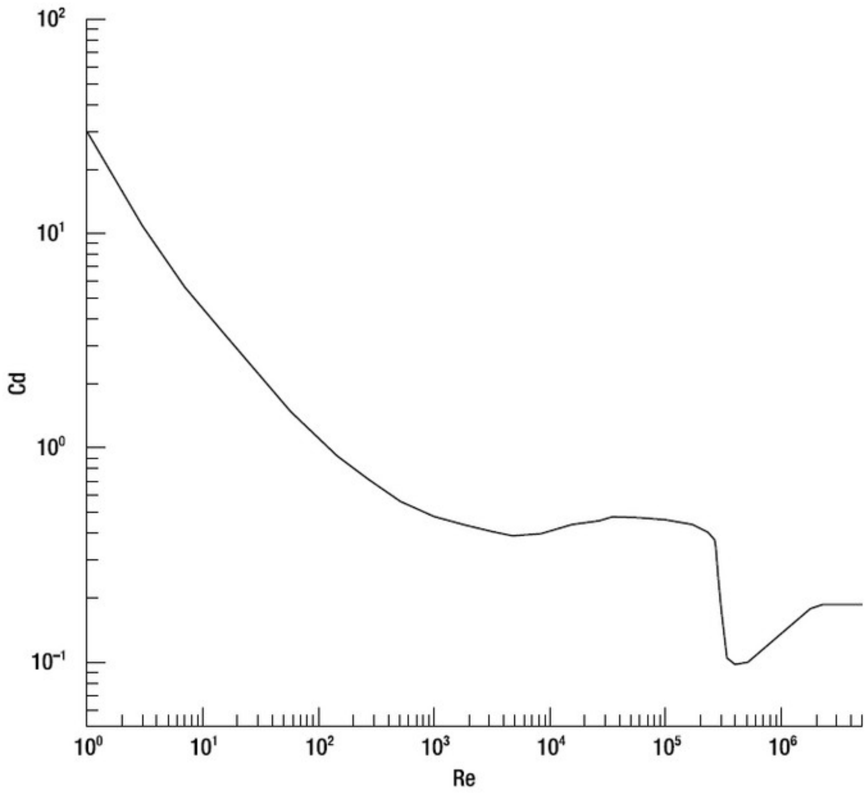
Shape	Picture	C_D
Square flat plate		1.17
Cube		1.05–1.07
Rotated cube		0.8–0.81
Solid hemisphere		0.42
60-degree cone		0.5
Sphere		0.4–0.47
2:1 Ellipsoid		0.27
Hollow hemisphere		1.4
Hollow hemisphere		0.38–0.4

Projectiles

- **Aerodynamic drag**
  - **Drag Coefficient**
    - **Laminar and Turbulent Flow**

$$C_d = C_d(\mathrm{Re})$$

$$\mathrm{Re} = \frac{\rho v L}{\mu}$$



The drag coefficient of a sphere as a function of Reynolds number

Projectiles

- **Aerodynamic drag**
  - **Drag Coefficient**
    - **Laminar and Turbulent Flow**

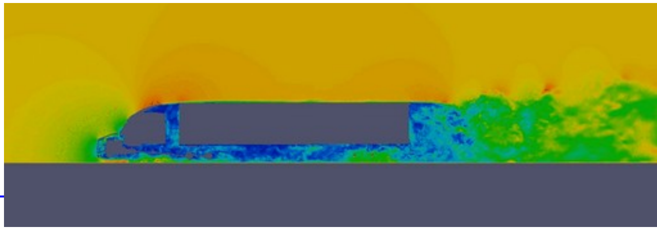
$$C_D = C_D(\mathrm{Re})$$

$$\mathrm{Re} = \frac{\rho v L}{\mu}$$

Laminar and Turbulent Drag Coefficients

Shape	Laminar C_D	Turbulent C_D
Sphere	0.4–0.47	0.2
2:1 Ellipsoid	0.27	0.13
Circular cylinder	1.2	0.3
2:1 Elliptical cylinder	0.6	0.2

Projectiles

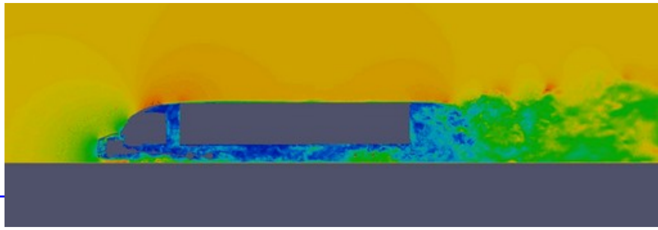


- **Aerodynamic drag**
  - **Drag Coefficient**
    - **Altitude Effects on Density**

Values of Air Density As a Function of Altitude

Altitude (*m*)	Altitude (*ft*)	Density (kg/m^3)	Density ($slug/ft^3$)
0.0	0.0	1.225	0.00238
305	1000	1.189	0.00231
610	2000	1.154	0.00224
914	3000	1.121	0.00218
1219	4000	1.088	0.00211
1524	5000	1.055	0.00205
2134	7000	0.992	0.00192
3048	10,000	0.905	0.00176

Projectiles



- **Aerodynamic drag**
  - **Equations of motion**

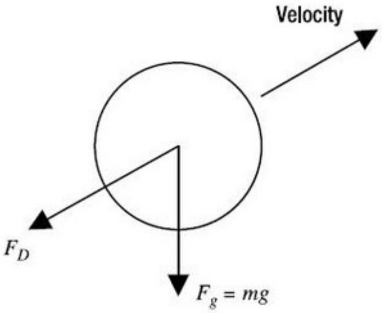
$$\vec{F} = m\,\vec{g} - |\vec{F}_D| \frac{\vec{v}}{|\vec{v}|}$$

$$F_d = C_d A \rho \frac{V^2}{2}$$

⇩

$$\ddot{\vec{r}} = \vec{g} - \frac{|\vec{F}_D|}{m} \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

⇩

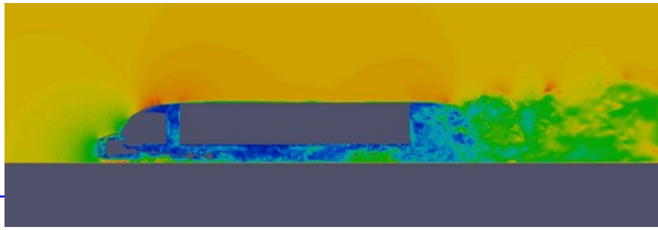


$$\frac{dv_x}{dt} = a_x = -\frac{F_D v_x}{mv}$$

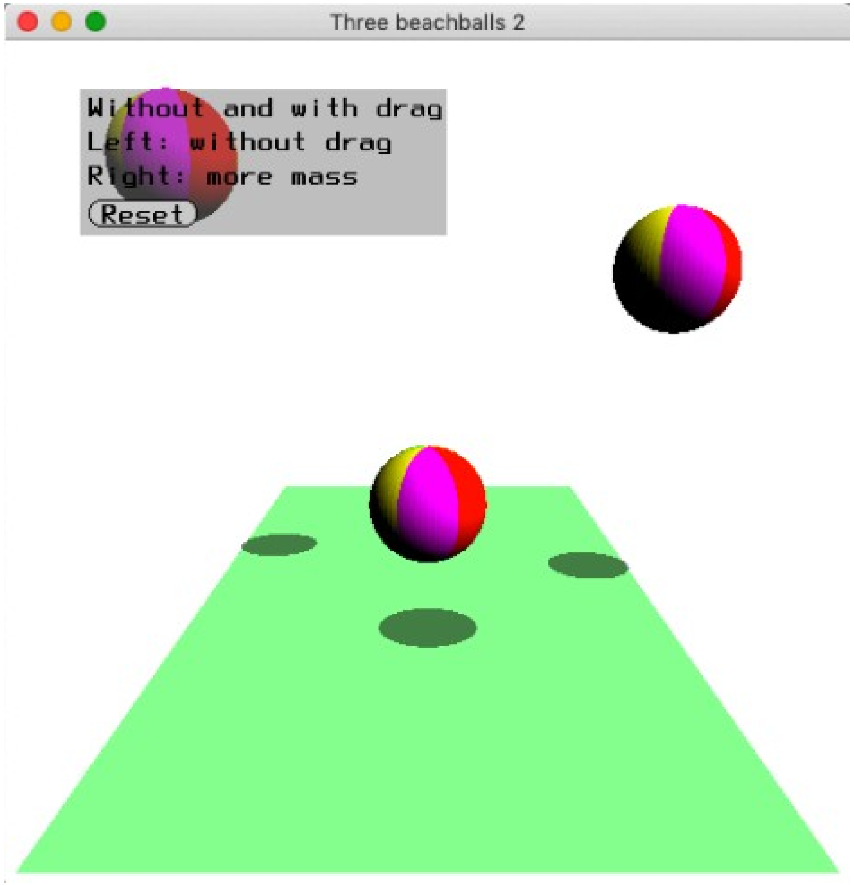
$$\frac{dv_y}{dt} = a_y = -\frac{F_D v_y}{mv}$$

$$\frac{dv_z}{dt} = a_z = -g - \frac{F_D v_z}{mv}$$

Projectiles



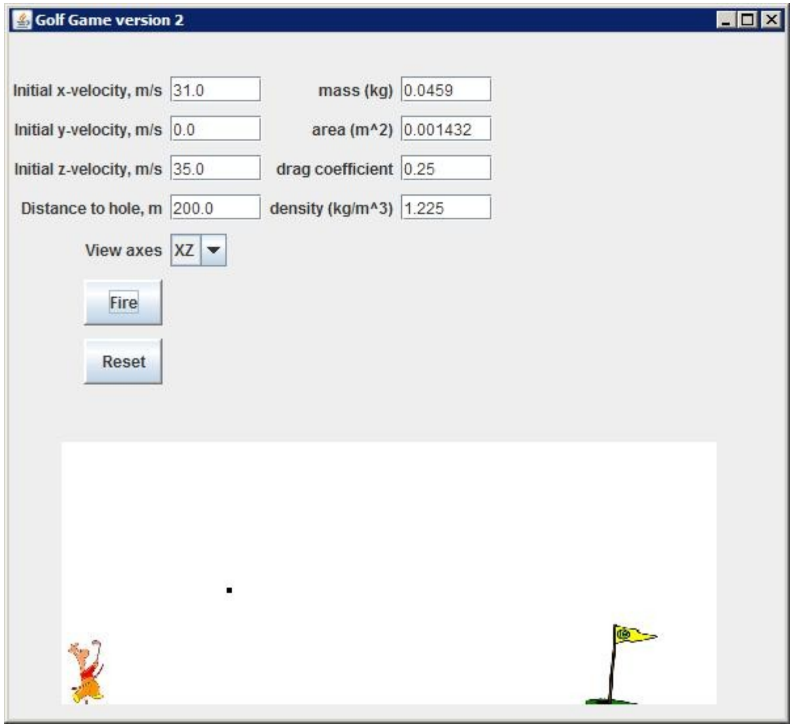
- **Aerodynamic drag**
  - **Three beachballs 2**



https://www.computer-graphics.se/demos/files03/Beachball-demos.zip

Projectiles

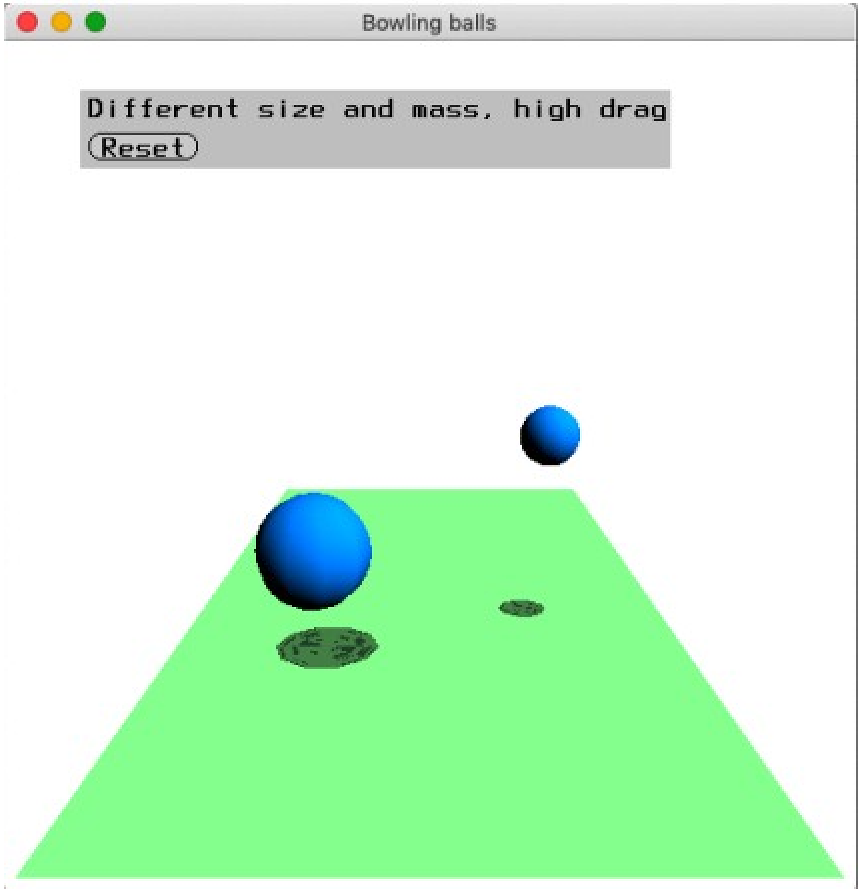
- **Aerodynamic drag**
  - **Golf Game Version 2**



https://ae.on.liu.se/tsbk03

Projectiles

- **Aerodynamic drag**
  - **Bowling balls**



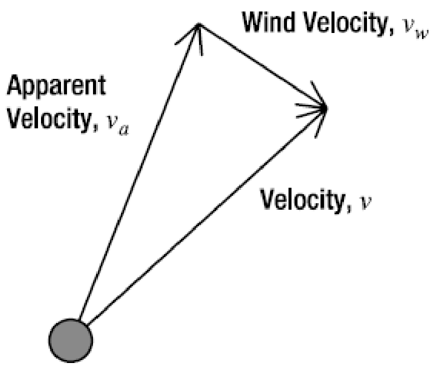
https://www.computer-graphics.se/demos/files03/Beachball-demos.zip

Projectiles

- **Aerodynamic drag**
  - **Summary**
    - Drag force acts in the opposite direction to the velocity. The magnitude of the drag force is proportional to the square of the velocity.
    - The three components of motion are coupled when drag is taken into account.
    - The drag force is a function of the projectile geometry.
    - The acceleration due to drag is inversely proportional to the mass of the projectile.
    - The drag on an object is proportional to the density of the fluid in which it is traveling.

Projectiles

- **Wind Effect**
  - **Equations of motion**



Apparent velocity is the vector sum of the projectile velocity and wind velocity.

$$\vec{F} = m\,\vec{g} - |\vec{F}_D| \frac{\vec{v}}{|\vec{v}|}$$

$$\Downarrow$$

$$\ddot{\vec{r}} = \vec{g} - \frac{|\vec{F}_D|}{m} \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|}$$

$$\Downarrow$$

$$\frac{dv_x}{dt} = a_x = -\frac{F_D v_x}{mv}$$

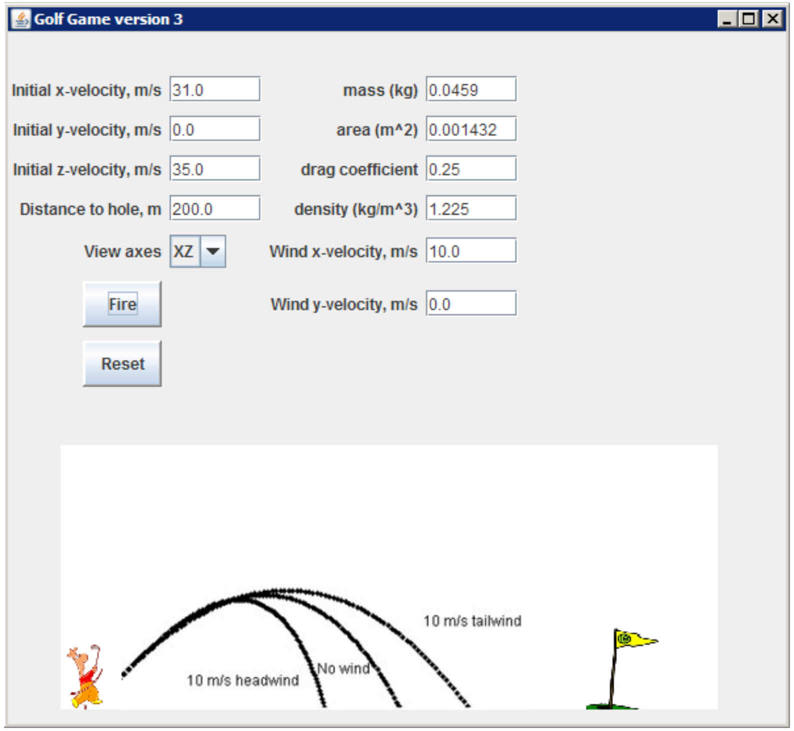
$$\frac{dv_y}{dt} = a_y = -\frac{F_D v_y}{mv}$$

$$\frac{dv_z}{dt} = a_z = -g - \frac{F_D v_z}{mv}$$

$$F_d = C_d A \rho \frac{V^2}{2}$$

Projectiles

- **Wind Effect**
  - **Golf Game Version 3**

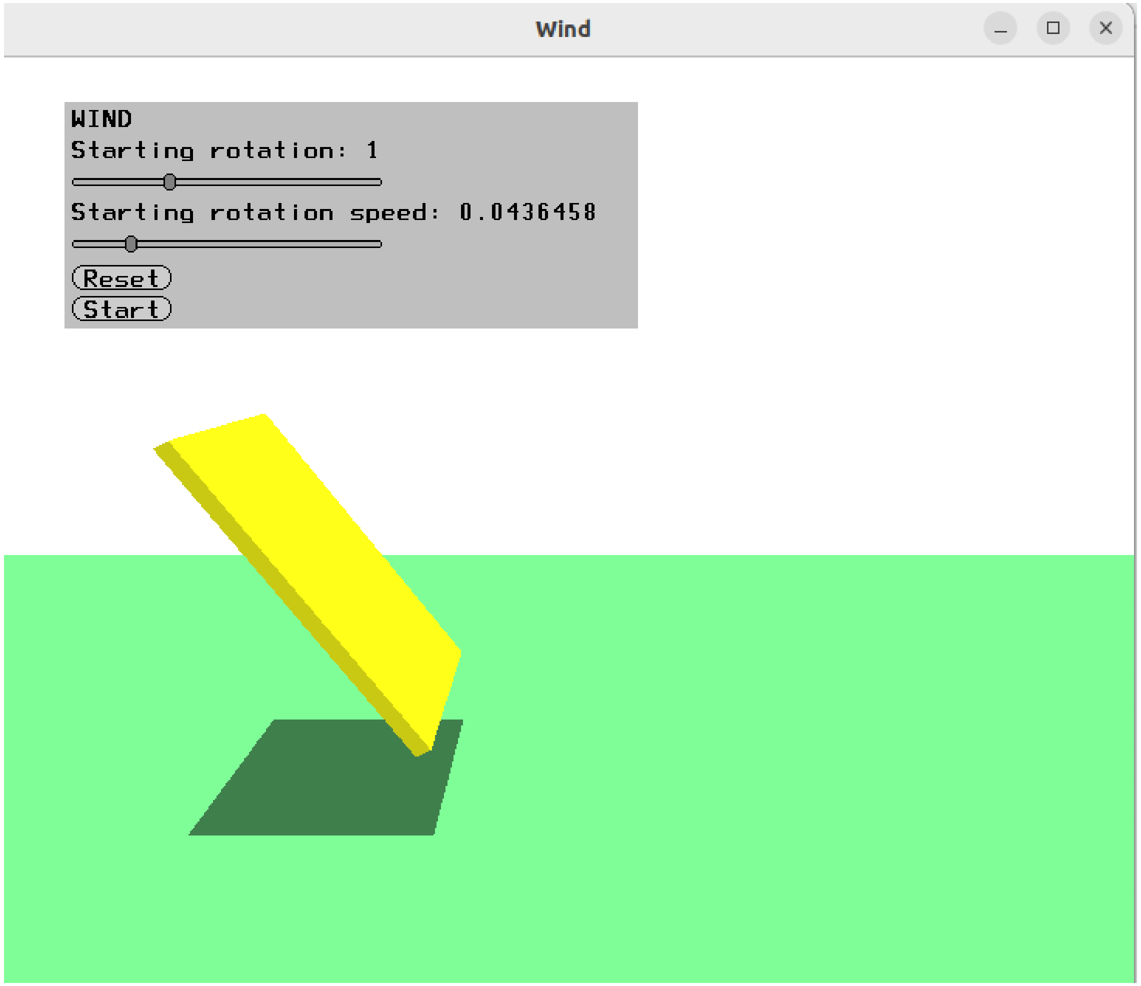


The effects of headwind or tailwind on a golf ball trajectory

…Java_Code\Chapter05_Projectile\GolfGame3.java (https://ae.onliu.se/tsbk03)

Projectiles

- **Wind Effect**



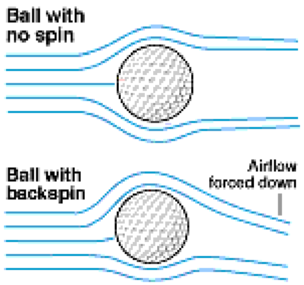
https://www.computer-graphics.se/demos/files03/Beachball-demos.zip

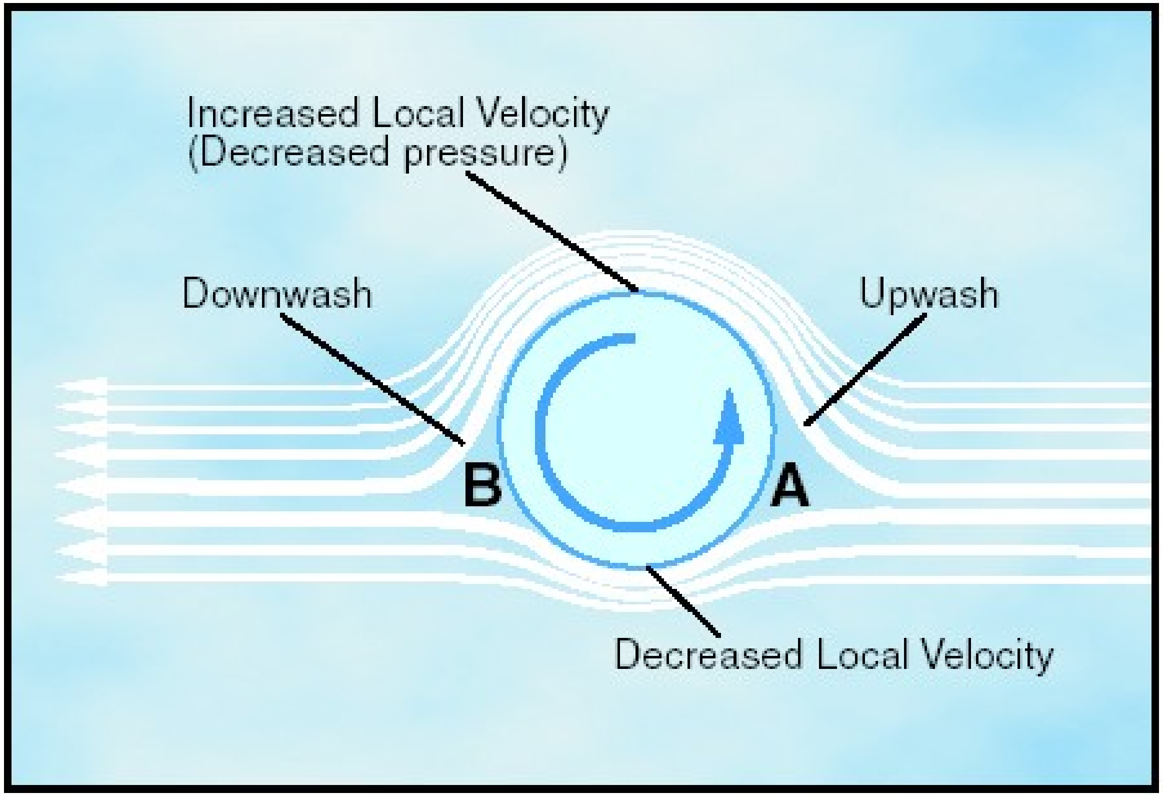
Projectiles

- **Wind Effect**
  - **Summary**
    - The presence of wind changes the apparent velocity seen by the projectile in flight. A headwind will increase the apparent velocity. A tailwind will decrease it.
    - The wind velocity affects the drag force in all three coordinate directions even if the wind velocities themselves are only in the x- and z-planes.

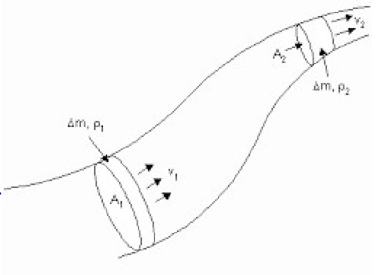
Projectiles

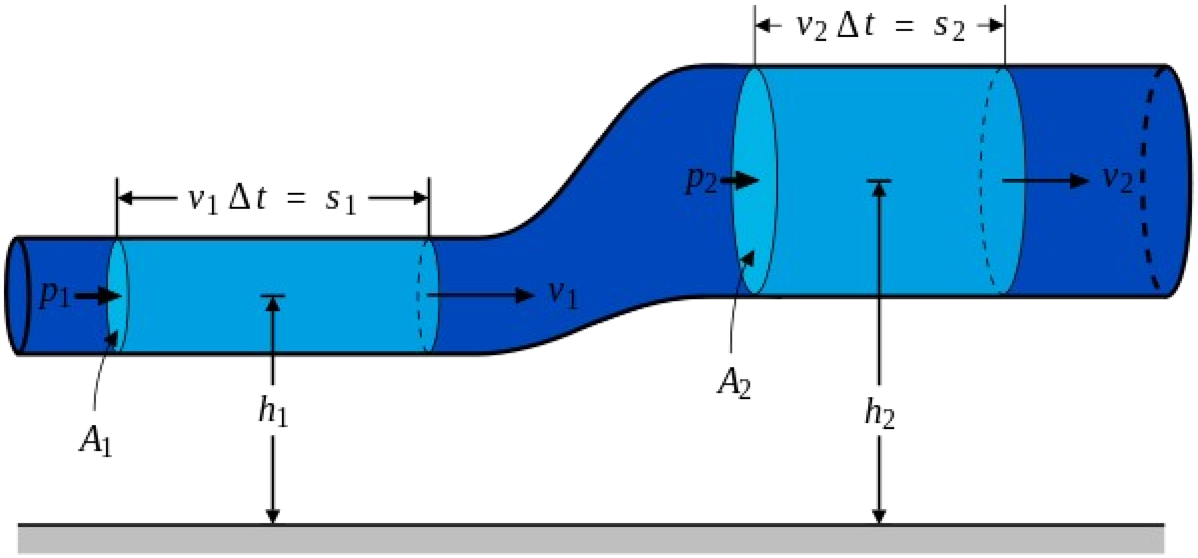
- **Projectiles**
  - **Spin Effect**





Bernoulli's equation

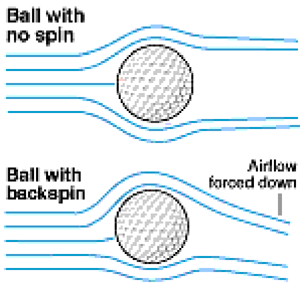




Bernoulli's equation: $p + \frac{1}{2}\rho v^2 + \rho g h = const$ ⇨ $p + \frac{1}{2}\rho v^2 = const$

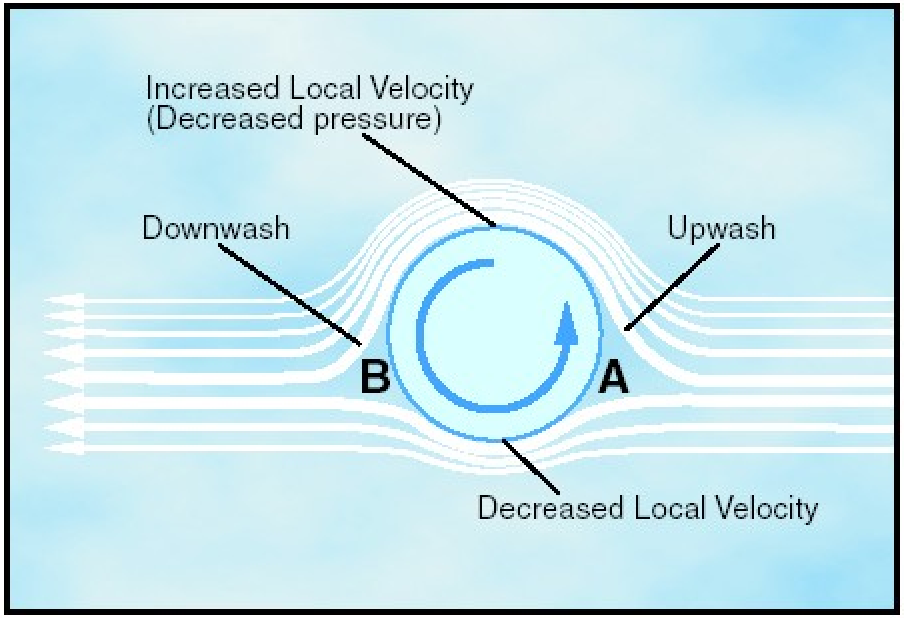
(*h=const*)

pressure **the fluid density** **velocity** **altitude**



Projectiles

- **Projectiles**
  - **Spin Effect**



⇨ **A spinning object generates lift**

Bernoulli's equation: $p+\frac{1}{2}\rho v^2+\rho gh=const$ ⇨ $p+\frac{1}{2}\rho v^2=const$

pressure (p), **the fluid density** (ρ), **velocity** (v), **altitude** (h)

$(h=const)$

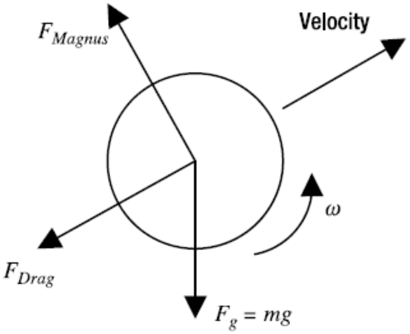
Projectiles

Spin Effect

- **Magnus force**

$$\vec{F}_M = C_L \rho \frac{v^2}{2} A \frac{[\vec{\omega} \times \vec{v}]}{|[\vec{\omega} \times \vec{v}]|}$$

The Magnus force lift coefficient



For a sphere: $C_L = \frac{r\omega}{v}$

For a cylinder: $C_L = \frac{2\pi r\omega}{v}$

- **Equation of motion**

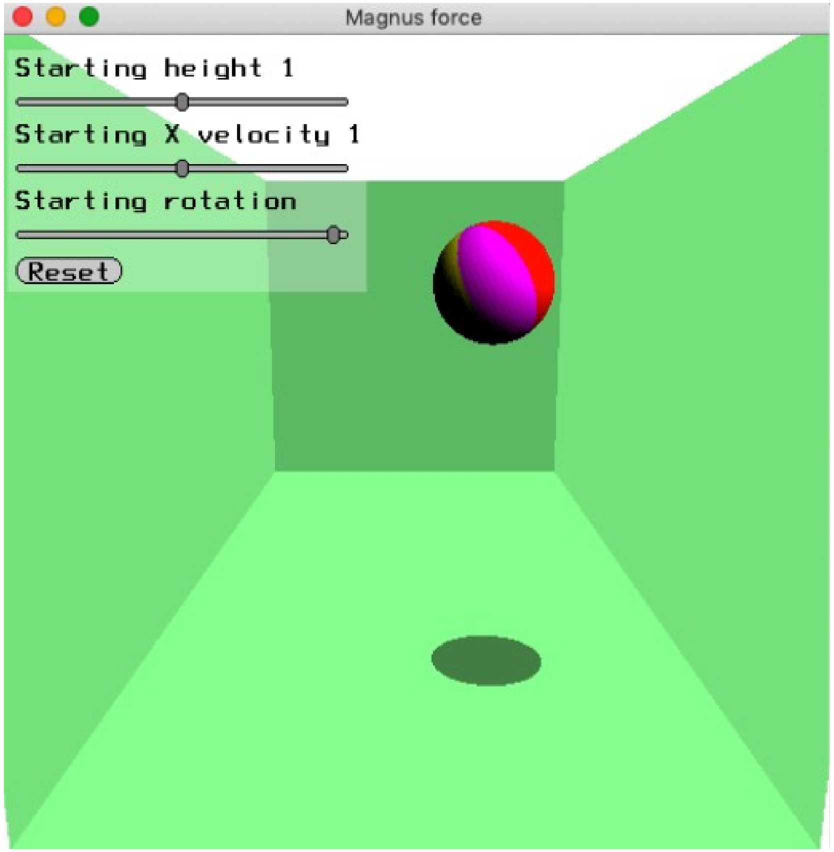
$$\vec{F} = m\,\vec{g} - |\vec{F}_D| \frac{\vec{v}}{|\vec{v}|} + C_L \rho \frac{v^2}{2} \frac{[\vec{\omega} \times \vec{v}]}{|[\vec{\omega} \times \vec{v}]|}$$

$$\Downarrow$$

$$\ddot{\vec{r}} = \vec{g} - \frac{|\vec{F}_D|}{m} \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} + C_L \rho \frac{|\dot{\vec{r}}|^2}{2m} \frac{[\dot{\vec{\theta}} \times \dot{\vec{r}}]}{|[\dot{\vec{\theta}} \times \dot{\vec{r}}]|}$$

Projectiles

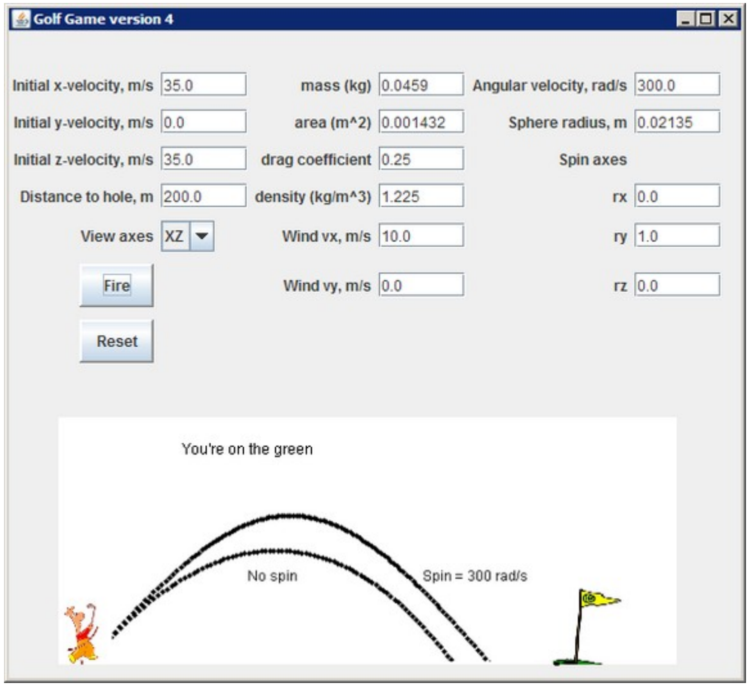
- **Spin Effect**
  - **Magnus force demo**



https://www.computer-graphics.se/demos/files03/Beachball-demos.zip

Projectiles

- **Spin Effect**
  - **Golf Game Version 4**



The effect of spin on golf ball flight

…Java_Code\Chapter05_Projectile\GolfGame4.java (htrps://ae.onliu.se/tsbk03)